TOPOLOGICAL MEANS

By

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CHAPTER I

INTRODUCTION

A topological groupoid is a Hausdorff space X equipped with a continuous multiplication, i.e., a function

$$X \times X \to X.$$

Unless expressed explicitly otherwise, the value of the function at (x,y) will be denoted by xy and if $A, B \subset X$, then we shall let

$$AB = \{ab \mid a \in A \text{ and } b \in B\}.$$

A semigroup is a topological groupoid in which associativity

(i) $\quad xy \cdot z = x \cdot yz$

holds for all x, y and z in the groupoid.

A mede is a topological groupoid M such that

(ii) $\quad xx = x$ for all $x \in M$

(iii) $\quad xy = yx$ for all $x, y \in M$.

A function $M \times M \to M$ satisfying both (ii) and (iii) is called a mean and hence we shall speak of the multiplication of a mede as the mean of the mede.

A topological groupoid is called medial if

(iv) $xy \cdot uv = xu \cdot yv$

holds for all x, y, u, v in the groupoid and is called autodistributive if

(v) $xy \cdot z = xz \cdot yz$ and $x \cdot yz = xy \cdot xz$

holds for all x, y, z in the groupoid. A groupoid satisfying (iv) is called entropic by Evans [8] and bisymmetric by Aczél [1], while the term medial is used by Stein [19]. We note that a topological groupoid satisfying (ii) and (iv) must also be autodistributive so that, in particular, a medial mede is autodistributive.

A function $f: G \to H$ between two topological groupoids G and H is called a homomorphism if

$$f(xy) = f(x)f(y)$$

for all x, y $\in$ G and in case this function is also a homeomorphism, it is called an iseomorphism and the groupoids G and H are said to be iseomorphic.

The study of medes has been centered primarily on two problems. The first is that of determining those spaces which admit the structure of a mede or, more generally, that of determining those spaces admitting an n-mean (see Chapter IV for definition) and the second is that of characterizing the arithmetic mean on the real line.

The notion of an n-mean was introduced by Aumann in [2] in which he studied the first problem showing, among other things, that for no $n \geq 2$ and $k \geq 1$ does the k-sphere admit an n-mean. Eckmann [6] extended this result in a most elegant fashion by showing that if a space X admits an n-mean, then its fundamental group is abelian, and for each $r \geq 1$ the mapping $x \to nx$ of either the homotopy group $\pi_r(X)$ or the integral singular homology group $H_r(X)$ into itself is an automorphism, the latter provided the space is pathwise connected. In Eckmann, Ganea and Hilton [7], the above work is put in the more general framework of categories giving rise to new results and also allowing the introduction of the dual notion of a comean. In particular, the first problem is solved in the class of compact polyhedra; for any $n \geq 2$, a compact polyhedron admits an n-mean iff it is contractible. The example of the n-adic solonoid discussed in Eckmann [6] shows that contractibility does not, in general, suffice to characterize those spaces which admit an n-mean. The general problem of determining those spaces admitting an n-mean remains quite open. For example, it is not known whether the space consisting of the curve $y = \sin \frac{1}{x}$, $0 < x \leq 1$, together with its limit continuum admits an n-mean for any $n \geq 2$. This is discussed in Chapter V.

Characterizations of the arithmetic mean on the real line have been known for some time. An early

characterization was given simultaneously by Kolmogoroff [14] and Nagumo [17], who defined a mean as a sequence of continuous functions

$$M_n : R^n \to R, \quad n = 1, 2, \ldots ,$$

each of which is symmetric, strictly increasing and idempotent ($M_n(x, \ldots ,x) = x$ for all $x \in R$) and which are connected by the condition that $M_n(x_1, \ldots ,x_n) = M_n(M_k(x_1, \ldots ,x_k), \ldots ,M_k(x_1, \ldots ,x_k), x_{k+1}, \ldots ,x_n)$ for all $x_1, \ldots ,x_n \in R$ and each $n = 1, 2, \ldots$. Here R denotes the real numbers. Kolmogoroff and Nagumo then showed that the above conditions are necessary and sufficient for the existence of a homeomorphism from R to R effecting, for each n, an iseomorphism between R under M_n and R under the n-ary arithmetic mean.

In 1948 Aczél [1] gave a simple characterization of the arithmetic mean in which mediality played the central role. Stated in the terminology which we have introduced, he showed that a (two-sided) cancellative medial mede on a connected subset of the real line is iseomorphic to a connected subset of the real line under arithmetic mean. Fuchs [9] showed that metrizability assumptions could be dispensed with and that it is enough to suppose that the underlying space is a connected totally ordered space. It was shown further by Hozzsú [13] that one may replace the assumption of mediality by that of autodistributivity.

That is, a cancellative, autodistributive mede on a connected totally ordered space is iseomorphic to a connected subset of the real line under arithmetic mean. The results of Aczél have been given a purely algebraic formulation by Evans [8]. Means on the real line are also discussed extensively by Hardy, Littlewood and Polya in [10].

If the map $x \to x \circ x$ of a commutative semigroup $(S,\circ)$ with unit u into itself is a homeomorphism and if one defines

$$\mu: S \times S \to S$$

by $\mu(x,y) = t(x \circ y)$, where t is the inverse of the map $x \to x \circ x$, then (S,μ) forms a medial mede such that the map $x \to \mu(x,u)$ is a homeomorphism. The main result of Chapter II asserts that every medial mede containing an element e such that the map $x \to xe$ is a homeomorphism can be obtained from some commutative semigroup with unit in this way.

Using the characterization obtained in Chapter II of all medial medes containing an element e such that the map $x \to xe$ is a homeomorphism, we are able in Chapter III to describe all such medes on an arc and on a triod.

In Chapter IV we discuss the homomorphisms of

those medes which are related to semigroups in the manner described above. In particular, it is shown that if each of the semigroups has a unit and if the image semigroup is algebraically a group, then every mede homomorphism can be realized as a translation of a semigroup homomorphism. This is a generalization of the elementary result for the real line (see, for instance, Hardy, Littlewood and Polya [10, page 66]) and is actually stated in terms of n-means.

As stated earlier, it is not known whether the curve $y = \sin \frac{1}{x}$, $0 < x \leq 1$, together with its limit continuum admits the structure of a mede. In Chapter IV we present some results concerning the admissibility of a mede on this space and on spaces similar to it.

CHAPTER II

MEANS IN SEMIGROUPS

In this chapter we obtain a characterization in terms of semigroups of all medial medes which contain an element e such that the map $x \to xe$ is a homeomorphism. This will be used later in obtaining information about medes from that known for semigroups. Some of the methods we employ are similar to those used by Evans in [8].

If G is a topological groupoid under each of two multiplications o and μ, then o,μ is said to form a medial pair provided

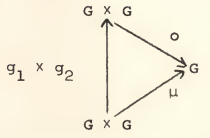
$$(*) \qquad (x \circ y) \mu (u \circ v) = (x \mu u) \circ (y \mu v)$$

for all $x, y, u, v \in G$. It should be observed that o,μ forms a medial pair iff μ is a homomorphism of the product groupoid $(G,o) \times (G,o)$ into (G,o) which, due to the symmetry present in (*), is in turn equivalent to o being a homomorphism of the product groupoid $(G,\mu) \times (G,u)$ into (G,μ). It is also of interest to note that a topological groupoid (G,o) is medial iff o,o forms a medial pair.

(2.1) Lemma. Let (G,o) be a topological groupoid, let g_1, g_2: $G \to G$ be commuting iseomorphisms of (G,o) onto itself and suppose that

$$\mu: G \times G \to G$$

is defined by the diagram



Then the following are equivalent:

(i) o is medial

(ii) μ, o forms a medial pair

(iii) μ is medial.

Proof. Writing xy for the value of μ at (x,y) and noting that g_1 and g_2 are onto, the equivalence of (i) and (ii) follows from the observation that

$$(x \circ y)(u \circ v) = g_1(x \circ y) \circ g_2(u \circ v)$$
$$= [g_1(x) \circ g_1(y)] \circ [g_2(x) \circ g_2(v)]$$

and

$$(xu) \circ (yv) = [g_1(x) \circ g_2(u)] \circ [g_1(y) \circ g_2(v)]$$

for all $x, y, u, v \in G$.

The equivalence of (ii) and (iii) is proved by

using the equivalence of (i) and (ii) and the symmetry present. More precisely, we know that

$$x \circ y = g_1^{-1}(x)g_2^{-1}(y)$$

for all $x, y \in G$ and that g_1^{-1} and g_2^{-1} are commuting iseomorphisms of $(G,\circ)$ onto itself. If we knew, in addition, that g_1^{-1} and g_2^{-1} were iseomorphisms of (G,μ) onto itself, then by reversing the roles of $\circ$ and μ and applying the equivalence of (i) and (ii), the equivalence of (ii) and (iii) would follow.

To show that g_1^{-1} and g_2^{-1} are, indeed, iseomorphisms of (G,μ) onto itself it suffices to show that each of g_1 and g_2 are such. But since g_1 and g_2 commute we have that

$$\begin{aligned} g_1(xy) &= g_1[g_1(x) \circ g_2(y)] \\ &= g_1g_1(x) \circ g_1g_2(y) \\ &= g_1g_1(x) \circ g_2g_1(y) \\ &= g_1(x)g_2(y) \end{aligned}$$

for all $x, y \in G$ and, similarly, that

$$g_2(xy) = g_2(x)g_2(y)$$

for all $x, y \in G$.

Mediality is sometimes used as a substitute for either commutativity or associativity. The following lemma lends credence to this.

(2.2) Lemma. A medial topological groupoid with unit is a commutative semigroup.

Proof. If u is the unit for a medial topological groupoid (G,o), then we have that, for all x, y, z $\in$ G,

$$x \circ y = (u \circ x) \circ (y \circ u)$$
$$= (u \circ y) \circ (x \circ u) = y \circ x$$

and

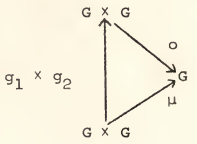
$$(x \circ y) \circ z = (x \circ y) \circ (u \circ z)$$
$$= (x \circ u) \circ (y \circ z) = x \circ (y \circ z).$$

The following lemma is immediate from lemmas (2.1) and (2.2).

(2.3) Lemma. Let (G,o) be a topological groupoid with unit, let g_1, g_2: G $\to$ G be commuting iseomorphisms of (G,o) onto itself, and suppose that

$$\mu: G \times G \to G$$

is defined by the diagram



Then the following are equivalent:

(i) (G,o) is a commutative semigroup

(ii) μ,o forms a medial pair

(iii) μ is medial.

If a is an element of a topological groupoid, then by the right translation by a we mean the mapping $x \to xa$ of the groupoid into itself. The left translation by a is defined similarly. An element e of a topological groupoid is called regular if both the right translation and the left translation by e are homeomorphisms of the groupoid onto itself and is called an idempotent if $e = ee$.

We now give a result which is essential to the proof of theorem (2.8), the main theorem of this chapter.

(2.4) Proposition. If a medial topological groupoid contains a regular idempotent e, then it admits the structure of a commutative semigroup having e as unit such that

$$xy = g_1(x) \circ g_2(y)$$

for all x, y in the groupoid, where o denotes the semigroup multiplication and where g_1 and g_2 are, respectively, the right and left translations in the groupoid by e. Furthermore, the groupoid multiplication and o form a medial pair.

Proof. We first note that g_1 and g_2 commute since

$$g_1g_2(x) = ex\cdot e = ex\cdot ee = ee\cdot xe = e\cdot xe = g_1g_2(x)$$

for all x in the groupoid G and that g_1 and g_2 are homomorphisms of G since

$$g_1(xy) = xy\cdot e = xy\cdot ee = xe\cdot ye = g_1(x)g_1(y)$$

and, similarly, $g_2(xy) = g_2(x)g_2(y)$ for all $x, y \in G$.

If we define $o: G \times G \to G$ by $x \circ y = g_1^{-1}(x)g_2^{-1}(y)$ we then have that

$$xy = g_1(x) \circ g_2(y)$$

for all $x, y \in G$ and that g_1 and g_2 are homomorphisms of (G,o) onto itself since they commute and are homomorphisms of G (see proof of (2.1)). Furthermore, e is a unit for (G,o) since

$$x \circ e = g_1^{-1}(x)g_2^{-1}(e) = g_1^{-1}(x)e = g_1g_1^{-1}(x) = x$$

and, similarly, $e \circ x = x$ for all $x \in G$. It then follows from lemma (2.3) that (G,o) is a commutative semigroup and that the groupoid multiplication and o form a medial pair.

The next lemma gives the canonical way for obtaining a mede from a commutative topological groupoid in which the map $x \to x \circ x$ is a homeomorphism and asserts further that if the topological groupoid is a semigroup, then the mede so obtained must be medial. This is discussed further in the

remarks preceding theorem (2.8).

(2.5) Lemma. If (G,o) is a commutative topological groupoid with unit such that the mapping $s: G \to G$ defined by $s(x) = x \circ x$ is a homeomorphism, then G forms a mede under the operation $\mu: G \times G \to G$ given by $\mu = s^{-1} \circ$ and the following are equivalent:

(i) o is associative

(ii) μ,o forms a medial pair

(iii) μ is medial.

Proof. It is clear that G forms a mede under μ and, in view of Lemma (2.3) (with $g_1 = g_2 = s^{-1}$), to show the equivalence of the three conditions it suffices to show that under each of (i), (ii) and (iii), s is a homomorphism of (G,o) onto itself.

For convenience, we denote the value of μ at (x,y) by xy.

(i) If o is commutative and associative then $s(x \circ y) = (x \circ y) \circ (x \circ y) = (x \circ x) \circ (y \circ y) = s(x) \circ s(y)$ for all $x, y \in G$.

(ii) For $x, y \in G$, we have that

$$s(xy) = xy \circ xy = (x \circ x)(y \circ y) = s(x)s(y)$$

and hence that

$$s(x \circ y) = ss(xy) = s[s(x)s(y)] = s(x) \circ s(y)$$

for all $x, y \in G$.

(iii) If we let e denote the unit for (G,o), then s^{-1} is given by

$$s^{-1}(x) = s^{-1}(x \circ e) = xe$$

so that since μ is medial we have that

$$s^{-1}(xy) = xy \cdot e = xy \cdot ee = xe \cdot ye = s^{-1}(x)s^{-1}(y)$$

and hence that

$$s(xy) = s(x)s(y)$$

for all $x, y \in G$. It then follows that

$$s(x \circ y) = ss(xy) = s[s(x)s(y)] = s(x) \circ s(y)$$

for all $x, y \in G$.

It may be remarked that, in the proof of lemma (2.5), the assumption that (G,o) has a unit was not used in showing that, under conditions (i) and (ii), s is a homomorphism of (G,o) so that, given all the hypotheses of the lemma except the existence of a unit for (G,o), we may conclude from (2.1) (a commutative and associative operation is medial) that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

These implications cannot be reversed. The first cannot since, for example, if o were a non-associative medial mean on a space G (for instance, a real number

interval under arithmetic mean), then the function s of the lemma would be the identity map on G so that the induced mean would coincide with o and hence would form, with o, a medial pair since o is medial. To show that the second implication cannot be reversed we define o on the real number interval [0,1] by

$$x \circ y = \max \{1 - x, 1 - y\}$$

so that the function s of the lemma is the homeomorphism given by $s(x) = 1 - x$. It follows that the induced mean is given by

$$xy = \min \{x, y\}$$

which is certainly medial but does not form, with o, a medial pair since, for example, $(01) \circ (01) = 0 \circ 0 = 1$ while $(0 \circ 0)(1 \circ 1) = 10 = 0$.

The following theorem gives a characterization of those semigroups with unit admitting the structure of a mede the mean of which forms, with the semigroup multiplication, a medial pair. If follows from a result in Eckmann, Ganea and Hilton [7] (also see Eckmann [6]) but a simple direct proof is in order.

(2.6) Theorem. A semigroup (S,o) with unit admits a mean which with o forms a medial pair iff it is commutative

and the mapping $s: S \to S$ given by $s(x) = x \circ x$ is a homeomorphism. Furthermore, if $(S,\circ)$ admits such a mean μ, then it is medial, unique and is given by

$$x \mu y = s^{-1}(x \circ y).$$

Proof. Suppose that $(S,\circ)$ admits a mean whose value at (x,y) we denote by xy, which with $\circ$ forms a medial pair and define $t: S \to S$ by $t(x) = xe$, where e is the unit of $(S,\circ)$. t is a homomorphism of $(S,\circ)$ into itself for if $x, y \in S$, then

$$\begin{aligned} t(x \circ y) &= (x \circ y)e = (x \circ y)(e \circ e) \\ &= (xe) \circ (ye) = t(x) \circ t(y) \end{aligned}$$

so that, in particular,

$$ts(x) = t(x \circ x) = t(x) \circ t(x) = st(x)$$

for all $x \in S$. If we note that for $x, y \in S$,

$$\begin{aligned} xy &= (x \circ e)(e \circ y) = (xe) \circ (ey) \\ &= t(x) \circ t(y) = t(x \circ y) \end{aligned}$$

we then have that

$$ts(x) = t(x \circ x) = xx = x$$

so that t is a continuous inverse for s and hence s is a homeomorphism. It also follows that $\circ$, given by $x \circ y = s(xy)$, is commutative since the mean is and that the

mean is given by

$$xy = s^{-1}(x \circ y)$$

and hence is unique. The mediality of the mean follows from lemma (2.5). The converse is immediate from lemma (2.5).

Preserving the notation of the preceding proof, if we note that the right translation by e in the mede (S,u) is t which is the inverse of the homeomorphism s, we see that the unit e of (S,o) is a regular element of the associated mede (S,μ). Also, since s and t are inverses, their fixed point sets coincide and hence, since the fixed points of s are precisely the idempotents of (S,o), then an element of S is an idempotent of (S,o) iff it is a fixed point of the right translation in the mede by e. We summarize this for later reference in the following

(2.7) Remark. If a semigroup (S,o) with unit e admits a mede whose mean forms with o a medial pair, then

(i) e is a regular element of the mede.

(ii) An element of S is an idempotent of (S,o) iff it is a fixed point of the right translation in the mede by e.

Theorem (2.6) asserts that if (S,o) is a commutative semigroup with unit having the property that the map

$x \to x \circ x$ is a homeomorphism of S onto S, then one obtains a medial mede (S,μ) by defining

$$x \mu y = t(x \circ y)$$

for all $x, y \in S$, where t is the inverse of the map $x \to x \circ x$. The remark (1.6 (i)) says, further, that the unit of the semigroup is a regular element of the mede. The following theorem asserts that, conversely, every medial mede containing a regular element is obtained from a commutative semigroup with unit in this way. We thus obtain a characterization in terms of semigroups of all medial medes containing a regular element.

(2.8) Theorem. If a medial mede contains a regular element e then it admits the structure of a commutative semigroup having e as unit such that the mean of the mede and the semigroup multiplication form a medial pair. Furthermore,

$$xy = t(x \circ y)$$

for all x, y in the mede, where o denotes the semigroup multiplication and t is the inverse of the map $x \to x \circ x$ of the mede into itself.

Proof. It follows from proposition (2.4) that the medial mede M admits a semigroup (M,o) having e as unit and such that the mean of the mede and o form a medial pair. The

second part of the theorem is immediate from uniqueness assertion in theorem (2.6).

The role of the identity of the semigroup in the preceding characterization seems to be essential. From the remarks following lemma (2.5) it is seen that if (S,o) is a commutative semigroup even without a unit but having the property that the map $x \to x \circ x$ is a homeomorphism, then one obtains a medial mede (S,μ) by defining

$$x \mu y = t(x \circ y)$$

for all $x, y \in S$, where t is the inverse of the map $x \to x \circ x$. It might then seem feasible that one could obtain all medial medes in this way. That this is not the case is seen by noting that the real number interval $[0,1]$ under arithmetic mean, a medial mede, cannot be obtained from a commutative semigroup in this way. For a commutative semigroup on the unit interval must contain a zero element z (see Paalman - de Miranda [18], proof of 2.5.6) and if a mede is obtained from the semigroup in the above manner, then z must also be a zero of the mede since, in this case,

$$x \mu z = t(x \circ z) = t(z) = z$$

for all x in $[o,1]$, where the last equality follows from the fact that t is the inverse of the map $x \to x \circ x$ which has z as a fixed point. However, the interval $[0,1]$

under arithmetic mean does not contain a zero.

This example suggests that possibly the correct question is the following: Can any medial mede be imbedded in a mede which can be obtained from a commutative semigroup in the above canonical way? The answer, so far as the author is aware, is not known.

CHAPTER III

MEDIAL MEDES ON AN ARC AND A TRIOD

In Chapter II we showed that all medial medes containing a regular element on a space can be obtained in a canonical way from certain commutative semigroups with unit on the space. Hence, if one knows enough about the commutative semigroups with unit on the space, then one can determine all medial medes with regular element on the space. In this chapter we are able to describe all medial medes with regular element on an arc by relying on a description due to Clifford [3] of a certain class of semigroups on an arc.

We first outline two methods of constructing a new semigroup from two given semigroups.

(3.1) Construction. If S is a semigroup with zero z and T is a semigroup with unit u, then $T \times S$ is a semigroup and it is easy to see that $(T \times \{z\}) \cup (\{u\} \times S)$ is a subsemigroup of $T \times S$. We will call the semigroup $(T \times \{z\}) \cup (\{u\} \times S)$ the ordinal sum of S and T. It is clear that each of $T \times \{z\}$ and $\{u\} \times S$ is a subsemigroup of the ordinal sum of S and T, the first being iseomorphic

to T and the second iseomorphic to S, and that their intersection consists of the single point (u,z) which acts as a zero for $\{u\} \times S$ and as a unit for $T \times \{z\}$. Indeed, if $x \in T$ and $y \in S$, then $(x,z)(u,y) = (x,z)$ so that each element of $\{u\} \times S$ acts as a unit for $T \times \{z\}$. Hence, the ordinal sum of S and T may be loosely described as the semigroup obtained by identifying z and u and requiring that each element of S act as a unit for T.

(3.2) Construction. Let S and T be semigroups with zeros z and z', respectively, and let $\varphi: S \to T$ be a continuous homomorphism such that $\varphi(z) = z'$. Then each of $\{z\} \times T$ and the graph $G(\varphi) = \{(s,\varphi(s) \mid s \in S\}$ of φ are subsemigroups of the product semigroup $S \times T$ as is their union $(\{z\} \times T) \cup G(\varphi)$ since, for $t \in T$ and $s \in S$, we have that

$$(z,t)(s,\varphi(s)) = (z,t\varphi(s)) \in \{z\} \times T$$

and

$$(s,\varphi(s))(z,t) = (z,\varphi(s)t) \in \{z\} \times T.$$

We will call the semigroup $(\{z\} \times T) \cup G(\varphi)$ the linear extension of T by φ. It is clear that the intersection of $\{z\} \times T$ and $G(\varphi)$ consists of the single point (z,z') and that $\{z\} \times T$ is iseomorphic to T. $G(\varphi)$ is iseomorphic to S as is seen from the function

$$g\colon S \to G(\varphi)$$

defined by $g(s) = (s,\varphi(s))$. The function g is a continuous homomorphism since

$$g = 1_S \times \varphi\colon S \to S \times T$$

and each of φ and the identity map 1_S on S is a continuous homomorphism, and g is also a homeomorphism since

$$\pi_1|_{G(\varphi)}\colon G(\varphi) \to S$$

is a continuous inverse of g, where $\pi_1\colon S \times T \to S$ is the usual first projection. Hence, the linear extension of T by φ may be loosely described as the semigroup obtained by identifying z and z' and requiring, for $t \in T$ and $s \in S$, that $ts = t\varphi(s)$ and $st = \varphi(s)t$.

An arc is a compact connected Hausdorff space with exactly two non-cutpoints. Such a space can always be totally ordered in such a way that the order topology coincides with the original topology and one non-cutpoint is a least element and the other non-cutpoint is a greatest element (see Hocking and Young [11], page 53). We shall always suppose an arc to have such a total order on it.

A standard thread is a semigroup whose underlying space is an arc and in which the least element is a zero and the greatest element is a unit of the semigroup.

Basic examples of a standard thread are the real number interval $[0,1]$ under ordinary multiplication (the unit thread), the real number interval $[1/2,1]$ with multiplication defined by $xy = \max\{1/2,$ ordinary product of x and $y\}$ (the nil thread), and an arc with multiplication given by $xy = \min\{x,y\}$ (a min thread). The following theorem shows that any standard thread can be constructed from these three semigroups. It was proven for standard threads on a real interval by Mostert and Shields [16] and extended to the non-metric case by Clifford [3]. Semigroups on totally ordered connected spaces have also been studied by Storey [20].

(3.3) Theorem. If S is a standard thread and $E = \{x \in S \mid x^2 = x\}$ is the set of its idempotents, then E is closed and if $\langle e,f\rangle$ is a component of $S \setminus E$, then $[e,f]$ is either a unit thread or a nil thread. Furthermore, if there is an idempotent between elements x and y of S, then $xy = \min\{x,y\}$.

In a standard thread S we shall refer to the closure of a component of $S \setminus E$ as a ligament of the semigroup. Thus a ligament of a standard thread is always either a unit thread or a nil thread. One should also observe that if x is in a standard thread $S = [0,u]$, then $Sx = xS = [0,x]$.

We now construct a semigroup as follows: Let $[0,b]$ be a standard thread, let $[a,0']$ be the order dual of a standard thread or else degenerate, let $\varphi\colon [0,b] \to [a,0']$ be an onto homomorphism (it is then necessarily continuous and order preserving (see Clifford [3])), and let T be the linear extension of $[a,0']$ by φ. If we totally order T by giving each of $[a,0']$ and $[0,b]$ their inherited order and by defining $x \leq y$ for each $x \in [a,0']$ and $y \in [0,b]$, then T is a semigroup on an arc such that

(i) its endpoints are idempotent

(ii) it contains a unit element

(iii) it is commutative.

A semigroup whose underlying space is an arc and which satisfies conditions (i), (ii) and (iii) will be called a D-semigroup. We remark that an equivalent definition is obtained by replacing (iii) by

(iii)' it contains a zero

(see Paalman - de Miranda [18], page 87).

The following theorem which asserts that every D-semigroup can be obtained via the above construction was obtained by Clifford [3] and, for the case of D-semigroups on a real interval, independently by Cohen and Wade [4].

(3.4) Theorem. Every D-semigroup has the above structure or its order dual. More precisely, if T is a D-semigroup,

then, by passing to the order dual if necessary, $T = [e,u]$ with $e \leq 0 < u$ where u is the unit and 0 is the zero of T, $[0,u]$ is a standard thread, $[e,0]$ is the order dual of a standard thread or else degenerate, and T is the linear extension of $[e,0]$ by φ where $\varphi\colon [0,u] \to [e,0]$ is given by $\varphi(x) = ex$.

We now prove a theorem which gives a characterization in terms of D-semigroups of all medial medes on an arc which contain a regular element.

(3.5) Theorem. The mean of a medial mede which contains a regular element and whose underlying space is an arc forms, with the multiplication of some D-semigroup on the arc, a medial pair, and conversely.

Proof. From theorem (2.8) we have that the arc M admits a commutative semigroup (M,o) with unit u such that o and the mean of the mede forms a medial pair so that it only remains to show that (M,o) is a D-semigroup.

The unit u of (M,o) cannot be a cutpoint of M (see Wallace [21, page 57]) and hence must be one of the endpoints of M. Therefore the right translation in the mede by u fixes the endpoint u and since, from (2.9 (i)), it is a homeomorphism, it must also fix the other endpoint so that by (2.9 (ii)) the endpoints are idempotents of (M,o). Since we already have that (M,o) is commutative

and contains a unit, it follows that (M,o) is a D-semigroup.

Conversely, if a mean on a D-semigroup forms, with the semigroup multiplication, a medial pair, then it follows from theorem (2.6) that the mean is medial and from (2.9 (i)) that the unit of the semigroup is a regular element of the associated mede.

From theorem (2.6) we know that in any semigroup with unit which admits a mean forming a medial pair with the semigroup multiplication, the mapping sending x into x^2 is a homeomorphism. We record exactly which D-semigroups have this property in the following

(3.6) Remark. The mapping $g: T \to T$ given by $g(x) = x^2$ of a D-semigroup T onto itself is a homeomorphism iff each ligament of the two standard threads whose union is T is a unit thread.

Proof. Since each ligament is a subsemigroup of T, the function g maps each ligament into itself and fixes each idempotent so that g is one-to-one (and hence a homeomorphism since T is compact) iff its restriction to each ligament is one-to-one. But this occurs precisely when each ligament is a unit thread. For, on the one hand, the restriction of g to a ligament which is a unit thread is clearly one-to-one while, on the other hand, a ligament which is a nil thread contains an element different from

the zero of the nil thread whose square is that zero and hence the restriction of g to such a ligament is not one-to-one.

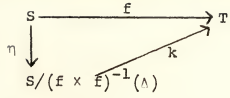
We state without proof two propositions which we will need and which are contained in Wallace [21, pages 30 and 33]. A subset I of a semigroup S is an ideal of S if sr, rs $\in$ I for each r $\in$ I and each s $\in$ S.

(3.7) Proposition. If S is a compact semigroup and if I is a closed ideal of S, then the relation

$$\mathfrak{C} = \Delta \cup (I \times I) \subset S \times S$$

is a congruence on S and the quotient $S/\mathfrak{C}$ forms a topological semigroup, where Δ denotes the diagonal of S x S. The quotient semigroup $S/\mathfrak{C}$ is usually denoted by S/I and is called the Rees quotient of S by I.

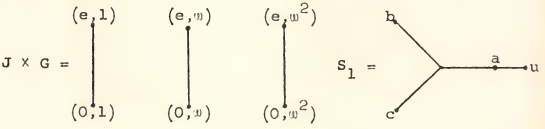
(3.8) Proposition. If S and T are compact semigroups, if f: S $\to$ T is a continuous onto homomorphism and if Δ is the diagonal of T x T, then $(f \times f)^{-1}(\Delta)$ is a closed congruence on S, $S/(f \times f)^{-1}(\Delta)$ is a semigroup such that the natural map η: S $\to$ $S/(f \times f)^{-1}(\Delta)$ is a homomorphism, and there is a unique iseomorphism k such that the diagram



is analytic.

We now turn our attention to describing all medial medes with regular element on a triod (defined below). We begin by constructing two semigroups.

If we let J = [0,e] be a standard thread and G = {1, ω, ω²} be the group of cube roots of 1, then J × G is a compact semigroup having {0} × G as a closed ideal so that by proposition (3.7), (J × G)/({0} × G) is a semigroup.



Let [0',u] be a standard thread and let S_1 be the ordinal sum of [0',u] and (J × G)/({0} × G). Then, writing a, b, c and 0 for (e,1), (e,ω), (e,ω²) and {0} × G, respectively, S_1 forms a semigroup such that

(i) u is a unit for S_1

(ii) $b^2 = c$ and $c^2 = b$

(iii) 0 is a zero for S_1

(iv) $bc = cb$.

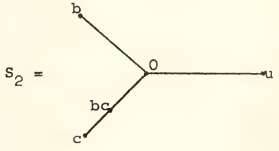
To construct the second semigroup, let $T = [b,u]$ be a D-semigroup with unit u whose zero 0 is different from b, let $J = [0',c]$ be a standard thread and let $\varphi: T \to J$ be a continuous homomorphism of T onto J. If S_2 is the linear extension of J by φ, then S is a semigroup such that

(i) u is a unit for S_2

(ii)' $b^2 = b$ and $c^2 = c$

(iii) 0 is a zero for S_2

(iv) $bc = cb$.

$S_2 =$ 

A space which is the union of arcs $A_1 = [a_1,u]$, $A_2 = [a_2,b]$ and $A_3 = [a_3,c]$ such that $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \{a_1\} = \{a_2\} = \{a_3\}$ will be called a <u>triod</u> and the point $a_1 = a_2 = a_3$ will be called the <u>center</u> of the triod. A semigroup whose underlying space is a triod will be called a Y_1-<u>semigroup</u> if it satisfies conditions (i), (ii), (iii) and (iv) above and will be called a Y_2-<u>semigroup</u> if it satisfies (i), (ii)', (iii) and (iv), where u, b and c are the endpoints and 0 is the center of the triod. The general problem of determining the structure

of a continuum semigroup irreducible over a finite set has been studied by Hunter [12].

It is convenient to state the following

(3.9) Lemma. If a semigroup S contains an element e such that $e^2 = e$ and $eS = S = Se$, then e is a unit for S. Proof. If $x \in S = eS$, then $x = ey$ for some $y \in S$ and hence $ex = e(ey) = e^2y = ey = x$. One shows $xe = x$ similarly.

The following pair of propositions shows that every Y_1- or Y_2-semigroup has the structure of the semigroups constructed above. The situation in proposition (3.10) is similar to the main result of Mostert and Shields in [16]. If x and y are elements of a triod, we denote by [x,y] the continuum irreducible between x and y.

(3.10) Proposition. Every Y_1-semigroup has the structure of the semigroup given in the first construction above. More precisely, if S is a Y_1-semigroup with endpoints b, c and u and with center 0, where u is the unit of S, then $bc \in [0,u]$, $S_0 = [0,b] \cup [0,c] \cup [0,bc]$ is a subsemigroup of S iseomorphic to $([0,bc] \times G)/(\{0\} \times G)$, where G is the group of cube roots of 1, and [bc,u] is a standard thread such that S is the ordinal sum of [bc,u] and S_0.

Proof. We let $A = [0,u]$, $B = [0,b]$ and $C = [0,c]$ and first prove a series of assertions.

(i) $Ab = B = bA$, $Ac = C = cA$, $bB = C = Bb$ and $cC = B = Cc$. We will prove only that $bA = B$, the proofs of the other equalities being similar. We observe that bA is connected and contains 0 and b so that $B = [0,b] \subset bA$. To show the reverse containment, suppose that $x \in A$ with $bx \in A \cup C$. Then $b, bx \in b[x,u]$ which is connected so that $0 \in [b,bx] \subset b[x,u]$ and hence $bp = 0$ for some $p \in [x,u]$. But then $x \in [0,p] = p[0,u]$ so that $x = pq$ for some $q \in [0,u]$ from which it follows that $bx = bpq = 0q = 0$. We have shown that for $x \in A$, $bx \in A \cup C$ implies that $bx = 0$ from which $bA \subset B$ follows.

(ii) A is a standard thread and hence $B^2 = C$, $C^2 = B$, $CA = AC = C$ and $BA = AB = B$. Since A is an arc with 0 as a zero and u as a unit, to show that A is a standard thread we have only to show that it is a subsemigroup of S. To this end suppose $x, y \in A$ such that $xy \in B \cup C$ with, say, $y \in [0,x]$. Then $0 \in [xy,x] \subset x[y,u]$, since $xy, x \in x[y,u]$ and $x[y,u]$ is connected, and hence $0 = xp$ for some $p \in [y,u]$. We then have that $y \in [0,p] = p[0,u]$ so that $y = pq$ for some $q \in [0,u]$ from which it follows that $xy = xpq = 0q = 0$. We therefore have that A is a subsemigroup of S. The other equalities follow from (i) and the fact that $A^2 = A$ as follows:

$$B^2 = AbAb = A^2b^2 = Ac = C.$$
$$C^2 = AcAc = A^2c^2 = Ab = B.$$
$$CA = cA^2 = cA = C = Ac = A^2c = AC.$$
$$BA = bA^2 = bA = B = Ab = A^2b = AB.$$

(iii) $BC = bC = Cb = cB = Bc = [0,bc] \subset A$.

From (i) we have immediately that

$$BC = AbAc = A^2bc = Abc,$$
$$bC = bAc = Abc = Acb = Cb \text{ and}$$
$$cB = cAb = Acb = Abc = Bc$$

so that all that remains is to show that $Abc = [0,bc]$ and that $bc \in A$. To see that $bc \in A$ one need only observe that bc is an idempotent, since $(bc)^2 = bcbc = b^2c^2 = cb = bc$, while the only idempotent in $B \cup C$ is 0, since $B^2 = C$ and $C^2 = B$. But then $Abc = [0,bc]$ since A is a standard thread and $bc \in A$.

(iv) $b[0,bc] = B = [0,bc]b$ and $c[0,bc] = C = [0,bc]c$. We will prove only that $b[0,bc] = B$, with the proofs of the other equalities being similar. We know that $B = [0,b] \subset b[0,bc]$, since 0 and $b = c^2 = b^2c = b(bc)$ are each in $b[0,bc]$ which is connected, while, on the other hand, we have that $b[0,bc] \subset bA = B$ and hence $b[0,bc] = B$.

(v) $S_0 = B \cup C \cup [0,bc]$ is a subsemigroup of S having bc as unit. That S_0 is a subsemigroup of S follows from (ii) and the fact that [0,bc] is itself a subsemigroup of S. To show that bc is a unit for S_0 we observe from (iii) and (iv) that

$$bcB = b[0,bc] = B = [0,bc]b = Bcb = Bbc,$$
$$bcC = cbC = c[0,bc] = C = [0,bc]c = Cbc \text{ and}$$
$$bc[0,bc] = bC = [0,bc] = Bc = [0,bc]bc$$

so that $bcS_0 = S_0 = S_0bc$. Hence, if we note that bc is an idempotent we have by lemma (3.9) that bc is a unit for S_0.

(vi) The functions

$$[0,bc] \to B \to C \to [0,bc],$$

each sending x into xb (x into bx), are homeomorphisms with their respective inverses being the functions

$$[0,bc] \leftarrow B \leftarrow C \leftarrow [0,bc],$$

each sending x into xc (x into cx). It follows from (i), (iii) and (iv) that each of the functions maps onto the proper range. That the mappings $x \to xb$ and $x \to xc$ are, in each case, inverses follows from the fact that bc is a unit for $S_0 = B \cup C \cup [0,bc]$, i.e., if $x \in S_0$, then $(xb)c = x(bc) = x$ and $(xc)b = x(cb) = x(bc) = x$.

We may introduce on each of the arcs A, B and C their natural total order by defining, for $x, y \in A(B,C)$, $x \leq y$ iff $x \in [0,y]$. Since each of the functions in (vi) is a homeomorphism fixing 0, we immediately obtain

(vii) Each of the functions in (vi) is order preserving.

(viii) $bx = xb$ and $cx = xc$ for all $x \in [0,bc]$. From (iv) we know that each of bx and xb is in B and we shall show $bx = xb$ by showing each of $bx < xb$ and $bx > xb$ to be false. For suppose $bx < xb$. Then from (vii) we have that $x = cbx < cxb$ and $bxc < xbc = x$, the first from premultiplication by c and the latter from post-multiplication by c. If we premultiply $bxc < x$ by b and then postmultiply by c we obtain $cxb = b^2xc^2 = b(bxc)c < bxc$. However, from the above we have that $bxc < x < cxb$, a contradiction. One shows $bx > xb$ is false in a similar way and hence $bx = xb$. That $cx = xc$ for all $x \in [0,bc]$ follows in a similar fashion.

(ix) $G = \{bc, b, c\}$ is a subgroup of S iseomorphic to the cube roots of 1. Since $b^2 = c$ and $b^3 = bb^2 = bc$, G must be a cyclic group of order 3 and any such is iseomorphic to the cube roots of 1.

We now turn to constructing the desired iseomorphism.

In doing so we may, in view of (ix), use G rather than the group of cube roots of 1, since it is more convenient to do so. Define the function

$$f\colon [0,bc] \times G \to S_0$$

by $f(x,t) = xt$. From (viii) we know that $xt = tx$ for each $x \in [0,bc]$ and each $t \in G$ so that f is a homomorphism;

$$f[(x_1,t_1)(x_2,t_2)] = f(x_1x_2,t_1t_2) = x_1x_2t_1t_2$$
$$= x_1t_1x_2t_2 = f(x_1,t_1)f(x_2,t_2).$$

From (vi) we have that f takes $[0,bc] \times bc$, $[0,bc] \times b$ and $[0,bc] \times c$ homeomorphically onto $[0,bc]$, B and C, respectively, so that f is onto and if $0 \neq y \in S_0$, then $f^{-1}(y)$ consists of one point while $f^{-1}(0)$ is the ideal $\{0\} \times G$ of $[0,bc] \times G$ and hence $(f \times f)^{-1}(\Delta_T)$ $= [(\{0\} \times G) \times (\{0\} \times G)] \cup \Delta_0$, where Δ_T is the diagonal of $T \times T$ and Δ_0 is the diagonal of $([0,bc] \times G) \times ([0,bc] \times G)$. Since each of the semigroups is compact, it then follows from lemma (3.8) that the induced map

$$\hat{f}\colon ([0,bc] \times G)/(\{0\} \times G) \to S_0$$

is an iseomorphism.

Since bc is an idempotent in the standard thread A we have that $[bc,u]$ is a standard thread and if $x \in S_0$ and $y \in [bc,u]$ then $xy = (xbc)y = x(bcy) = xbc = x$ and

$yx = y(bcx) = (ybc)x = bcx = x$ and hence S is the ordinal sum of $[bc,u]$ and S_0.

(3.11) Proposition. Every Y_2-semigroup has the structure of the semigroup in the second construction above. More precisely, if S is a Y_2-semigroup with center 0 and with endpoints b, c and u, where u is the unit of S, then $bc \in [b,c]$ and, supposing from the symmetry present that $bc \in [0,c]$, $[b,u]$ is a D-semigroup, $[0,c]$ is a standard thread and S is the linear extension of $[0,c]$ by φ, where $\varphi\colon [b,u] \to [0,c]$ is given by $\varphi(x) = xc$.

Proof. The proof is contained in the following series of assertions. It is convenient to let $A = [0,u]$, $B = [0,b]$ and $C = [a,c]$.

(i) $Ac = C = cA$ and $bA = B = Ab$. We will show only that $bA = B$, the proofs of the other equalities being similar. If $bx \in A \cup C$ with $x \in A$, then $0 \in [bx,b] \subset b[x,u]$ since $bxb \in b[x,u]$ and $b[x,u]$ is connected and hence $bp = 0$ for some $p \in [x,u]$. We then have that $x \in [0,p] = p[0,u]$ so that $x = pq$ for some $q \in [0,u]$ from which it follows that $bx = bpq = 0q = 0$. Therefore $bx \in B$ for all $x \in A$, i.e., $bA \subset B$. On the other hand, $0, b \in bA = b[0,u]$ and $b[0,u]$ is connected so that $B = [0,b] \subset bA$ and hence $bA = B$.

(ii) Each of A, B and C is a standard thread. To show $A^2 \subset A$ suppose that $xy \in B \cup C$ for some $x, y \in A$ with, say, $y \in [0,x]$. Then $0 \in [xy,x] \subset x[y,u]$, since $xy, x \in x[y,u]$ and $x[y,u]$ is connected, and hence $xp = 0$ for some $p \in [y,u]$. But then $y \in [0,p] \subset p[0,u]$ so $y = pq$ for some $q \in [0,u]$ and hence $xy = xpq = 0q = 0$. We have shown that if $xy \in B \cup C$ with $x, y \in A$, then $xy = 0$ from which it follows that $A^2 \subset A$. Using this fact and (i) we obtain

$$B^2 = AbAb = A^2b^2 \subset Ab = B \quad \text{and}$$
$$C^2 = AcAc = A^2c^2 \subset Ac = B$$

so that each of A, B and C are subsemigroups and each is an arc having 0 as an endpoint. In order to show that they are standard threads it only remains to show that in each case, the other endpoint is a unit for the semigroup. Clearly u is a unit for A and we also have that

$$bB \subset B = [0,b] \subset b[0,b] \subset bB,$$

i.e., $bB = B$, and in a similar fashion that $B = Bb$ and $cC = C = Cc$ so that applying lemma (3.9) we see that b is a unit for B and c is a unit for C.

(iii) $bc \in B \cup C$ and, supposing from the symmetry present that $bc \in C$, $Bc \subset C \supset cB$. If $bc \in A$, then $bc = b^2c = b(bc) \subset bA = B$ so that $bc \in A \cap B = \{0\}$ and hence $bc \in B \cup C$. If $bc \in C$, one shows that $Bc \subset C \supset cB$

in a manner entirely similar to that in (i).

(iv) [b,u] is a D-semigroup. It follows from (i) and (ii) that

$$AB = AAb = Ab = B = bA = bAA = BA$$

and since we know from (ii) that $A^2 \subset A$ and $B^2 \subset B$, it follows that $[b,u] = A \cup B$ is a subsemigroup of S. It is clear that [b,u] is an arc with idempotent endpoints and that u is a unit and 0 is a zero for [b,u] and hence it is a D-semigroup.

(v) The function φ: $[b,u] \to [0,c]$ defined by $\varphi(x) = xc$ is a continuous onto homomorphism. The continuity of the multiplication in the semigroup implies that φ is continuous and from (i) and (iii) we have that $Ac = C$ and $Bc \subset C$ so φ does indeed map $[b,u] = A \cup B$ onto $[0,c]$. To show that φ is a homomorphism we let $x, y \in [b,u]$ and note that $c(yc) = yc$, since c is a unit for C and $yc \in C$, and hence

$$\varphi(xy) = xyc = x[c(yc)] = (xc)(yc) = \varphi(x)\varphi(y).$$

(vi) S is the linear extension of [0,c] by φ. We first observe that if $x \in [b,u]$, then each of cx and xc is in C so that since c is a unit for C we have that

$$cx = (cx)c = c(xc) = xc$$

for all $x \in [b,u]$. Then if $x \in [b,u]$ and $y \in [0,c]$ we have that

$$xy = x(cy) = (xc)y = \varphi(x)y$$

and that

$$yx = (yc)x = y(cx) = y(xc) = y\varphi(x)$$

so that S is the linear extension of $[0,c]$ by φ.

In the next theorem we obtain a characterization in terms of Y_1- and Y_2-semigroups of all medial medes on a triod which contain a regular element and in which the zero of the mede is the center of the triod.

(3.12) Theorem. If M is a medial mede containing a regular element and whose underlying space is a triod and if the center of the triod is a zero for M, then its mean forms a medial pair with the multiplication of either a Y_1-semigroup or a Y_2-semigroup on the space.

Proof. If u denotes a regular element of M then from (2.8) we have that M admits the structure of a commutative semigroup (M,o) having u as unit and such that the mean of the mede and o form a medial pair so that it only remains to show that (M,o) is either a Y_1-semigroup or a Y_2-semigroup.

We know from (2.6) that the function $s: M \to M$ defined by $s(x) = x \circ x$ must be a homeomorphism and that

the mean of the mede is given by

$$xy = s^{-1}(x \circ y)$$

so that, in particular, the right translation in the mede by u is given by s^{-1} since

$$s^{-1}(x) = s^{-1}(x \circ u) = xu,$$

and the center of the triod, which we denote by 0, is a zero of the semigroup as well as of the mede since

$$x \circ 0 = s^{-1}(x0) = s^{-1}(0) = 0.$$

Since the unit u of (M,o) cannot be a cutpoint of M (see Wallace [21, page 57])it must be one of the endpoints of M and, since (M,o) is commutative, $b \circ c = c \circ b$, where b and c denote the other endpoints of M. We have shown that (M,o) satisfies conditions (i), (iii) and (iv) of the definition of a Y_1 (Y_2)-semigroup and we now show that one of conditions (ii) and (ii)' must hold.

From the fact that s^{-1} is a homeomorphism and that $s^{-1}(u) = u$ it follows that either $s^{-1}(b) = b$ and $s^{-1}(c) = c$, or $s^{-1}(b) = c$ and $s^{-1}(c) = b$. If $s^{-1}(b) = b$ and $s^{-1}(c) = c$, then we have from (2.7) that each of b and c are idempotents of (M,o) and hence condition (ii)' is satisfied. If, on the other hand, $s^{-1}(b) = c$ and $s^{-1}(c) = b$, then $b \circ b = s(bb) = s(b) = c$ and

$c \circ c = s(cc) = s(c) = b$ so that condition (ii) is satisfied. Therefore $(M,\circ)$ is either a Y_1-semigroup or a Y_2-semigroup.

CHAPTER IV

HOMOMORPHISMS OF THE MEAN STRUCTURE IN SEMIGROUPS

In this chapter we introduce the notion of an n-mean on a space and in case such n-means are related to semigroup structures much as in chapter II, we are able to establish a connection between the homomorphisms of the n-mean structures on two spaces and the homomorphisms of the semigroup structures on the spaces.

An n-mean on a Hausdorff space X is a continuous function

$$\mu: X^n \to X$$

satisfying

(i) $\mu(x,x, \ldots ,x) = x$ for all $x \in X$.

(ii) $\mu(x_1,x_2, \ldots ,x_n) = \mu(x_{\sigma(1)},x_{\sigma(2)}, \ldots ,x_{\sigma(n)})$

for all $x_1, x_2, \ldots ,x_n \in X$ and all permutations σ of $\{1, 2, \ldots ,n\}$.

It is clear that a mean as defined in chapter II is simply a 2-mean. Since the only 1-mean on a space is the identity map, we will suppose always that $n \geq 2$. If S is a semigroup, then an n-mean on S is called morphic with respect to S if μ is a semigroup homomorphism of the product

semigroup S^n into S. In case n = 2, this is equivalent to μ and the semigroup multiplication forming a medial pair. In this chapter it is convenient to write all semigroups additively so that, in particular, if a semigroup has an identity, we denote it by 0.

The following theorem, which follows from a result in Eckmann, Ganea and Hilton [7] (see also Eckmann [6]), is essentially a restatement of theorem (2.6) for n-means.

(4.1) Theorem. A semigroup S with unit admits a morphic n-mean iff it is commutative and the mapping $s: S \to S$ defined by $s(x) = nx$ is a homeomorphism. When this is the case, it is unique and is given by

$$\mu(x_1, \ldots, x_n) = s^{-1}(x_1 + \ldots + x_n).$$

In particular, one should observe that if the semigroup S admits a morphic n-mean μ, then

$$n\mu(x_1, \ldots, x_n) = x_1 + \ldots + x_n$$

for all $x_1, \ldots, x_n \in S$ and S is uniquely n-divisible so that nx = ny implies that x = y.

If μ is an n-mean on a space X and if ν is an n-mean on a space Y, then a function $f: X \to Y$ is called

a homomorphism of the n-mean structures on the spaces if

$$f\mu(x_1, \ldots, x_n) = \nu(f(x_1), \ldots, f(x_n))$$

for all $x_1, \ldots, x_n \in X$. If we denote by $f^{\#}$ the function from X^n to Y^n defined by

$$f^{\#}(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n)),$$

then the above equality may be written as $f\mu = \nu f^{\#}$.

The next lemma tells us that in the case of morphic n-means on semigroups, a translation of a semigroup homomorphism must be a homomorphism of the n-mean structures.

(4.2) Lemma. Suppose that S and T are semigroups with unit, each admitting a morphic n-mean, and suppose that

$$g: S \to T$$

is a semigroup homomorphism. Then for any $b \in T$ the function

$$f: S \to T$$

defined by $f(x) = g(x) + b$ is a homomorphism of the n-mean structures on S and T.

Proof. Let μ denote the morphic n-mean on S and ν denote the morphic n-mean on T. Since g is a semigroup homomorphism, then from the fact that

$$\begin{aligned} n g\mu(x_1, \ldots, x_n) &= g n\mu(x_1, \ldots, x_n) \\ &= g(x_1 + \ldots + x_n) \\ &= g(x_1) + \ldots + g(x_n) \\ &= n\nu g^{\#}(x_1, \ldots, x_n) \end{aligned}$$

and that the semigroup S is uniquely n-divisible (see theorem (4.1)) we have that $\nu g^{\#} = g\mu$, i.e., g is a homomorphisms of the n-mean structures on S and T. Then, if $x_1, \ldots, x_n \in S$ we have that

$$\begin{aligned} \nu f^{\#}(x_1, \ldots, x_n) &= \nu(g(x_1) + b, \ldots, g(x_n) + b) \\ &= \nu[(g(x_1), \ldots, g(x_n)) \\ &\quad + (b, \ldots, b)] \\ &= \nu g^{\#}(x_1, \ldots, x_n) + \nu(b, \ldots, b) \\ &= g\mu(x_1, \ldots, x_n) + b \\ &= f\mu(x_1, \ldots, x_n) \end{aligned}$$

so that f is a homomorphism of the n-mean structures on S and T.

The preceding lemma showed that, on semigroups with unit, a translation of a semigroup homomorphism must be a homomorphism of any n-mean structures on the semigroups. The following theorem shows that in the event that the image semigroup is algebraically a group, every homomorphism of the n-mean structures can be realized in this way.

(4.3) Theorem. If each of the semigroups S and T has a unit and admits a morphic n-mean and if T is algebraically a group, then a continuous function $f: S \to T$ is a homomorphism of the n-mean structures iff there exists a continuous semigroup homomorphism $g: S \to T$ and an element $b \in T$ such that

$$f(x) = g(x) + b$$

for all $x \in S$.

Proof. Lemma (4.2) contains the "if" part of the theorem and in order to show the converse we suppose that $f: S \to T$ is a continuous homomorphism of the n-mean structures on S and T and let μ and ν denote the morphic n-means on S and T, respectively. We first note that

$$\begin{aligned}
&\mu(x_1 + \ldots + x_n, 0, \ldots, 0) \\
&= \mu(x_1, 0, \ldots, 0) + \ldots + \mu(x_n, 0, \ldots, 0) \\
&= \mu(x_1, 0, \ldots, 0) + \ldots + \mu(0, \ldots, 0, x_n) \\
&= \mu(x_1, \ldots, x_n)
\end{aligned}$$

for all $x_1, \ldots, x_n \in S$ so that

$$\begin{aligned}
f(x_1) + \ldots + f(x_n) &= n\nu f^{\#}(x_1, \ldots, x_n) \\
&= nf\mu(x_1, \ldots, x_n) \\
&= nf\mu(x_1 + \ldots + x_n, 0, \ldots, 0) \\
&= n\nu f^{\#}(x_1 + \ldots + x_n, 0, \ldots, 0) \\
&= f(x_1 + \ldots + x_n) \\
&\quad + (n - 1)f(0)
\end{aligned}$$

for all $x_1, \ldots, x_n \in S$. Since T is a group it contains an inverse $-f(0)$ for $f(0)$ and hence if we define $g: S \to T$ by

$$g(x) = f(x) + (-f(0))$$

then g is clearly continuous and it is straightforward to verify that $g(x_1 + \ldots + x_n) = g(x_1) + \ldots + g(x_n)$ for all $x_1, \ldots, x_n \in S$ so that, in particular, $g(x + y) = g(x) + g(y)$ for all $x, y \in S$ since $g(0) = f(0) + (-f(0)) = 0$. We then have that $f(x) = g(x) + b$ for all $x \in S$ with g a continuous semigroup homomorphism, where $b = f(0)$.

Observing that the above representation is independent of n, we obtain the following

(4.4) Corollary. If each of the semigroups S and T admits both a morphic n-mean and a morphic m-mean and if T is algebraically a group, then a continuous function from S into T is a homomorphism of the n-mean structures on S and T iff it is a homomorphism of their m-mean structures.

Before proving the next corollary of theorem (4.3) we prove an elementary lemma.

(4.5) Lemma. A continuous homomorphism between the additive groups of two real topological linear spaces is a linear transformation between the linear spaces. Proof. Supposing that B is a continuous homomorphism of the additive group of the real topological linear space L into the additive group of the real topological linear space M, we have only to show that B respects scalar multiplication, i.e., that $B(rx) = rB(x)$ for all $x \in L$ and all real numbers r.

From the fact that B is an additive homomorphism we have that $B(px) = pB(x)$ for all $x \in L$ and all integers p and, in particular, that $2B(\frac{1}{2}x) = B[2(\frac{1}{2}x)] = B(x)$ for all $x \in L$. It follows that $B(\frac{1}{2}x) = \frac{1}{2}B(x)$ for all $x \in L$ and, inductively, that $B(\frac{1}{2^k}x) = \frac{1}{2^k}B(x)$ for all $x \in L$ and all positive integers k and hence we have that

$$B(rx) = rB(x)$$

for all $x \in L$ and all dyadic rational numbers r. But for each $x \in L$, $\{r \mid r$ is real and $B(rx) = rB(x)\}$ is a closed subset of the real numbers and contains the dyadic rationals which are dense in the set of real numbers. Therefore $B(rx) = rB(x)$ for all $x \in L$ and all real numbers r.

Since, for any $n \geq 1$, the mapping $x \to nx$ of a

real topological linear space into itself is a homeomorphism, any such space admits a morphic n-mean for all n. Theorem (4.3) reduces to the following corollary in the case of real topological linear spaces. This generalizes the well-known result for the real line. An affine transformation of a linear space L into a linear space M is a function $A: L \to M$ such that

$$A(x) = B(x) + b$$

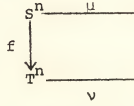
for all $x \in L$, where $b \in M$ and $B: L \to M$ is a linear transformation.

(4.6) Corollary. A continuous function between two real topological linear spaces is a homomorphism of the n-mean structures on the spaces iff it is an affine transformation.

Proof. If L and M are real topological linear spaces, then we know from theorem (4.3) that a continuous function $A: L \to M$ is a homomorphism of the n-mean structures on L and M iff $A(x) = B(x) + b$ for all $x \in L$ for some $b \in M$ and some continuous homomorphism of the additive groups of L and M. But it follows from lemma (4.5) that B is a continuous homomorphism of the additive groups of L and M iff it is a continuous linear transformation from L to M. The corollary follows.

We conclude this chapter with two propositions of

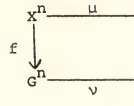
a slightly different character. In the preceding results we discussed functions $f_0: S \to T$ which were homomorphisms of some n-mean structures on S and T, i.e., such that the diagram



is analytic, where $f = f_0^{\#}$.

We now drop the condition that $f = f_0^{\#}$ and, for a given such function f_0, give a characterization in a more general setting than before of all those functions f making the above diagram analytic. A topological group is a semigroup which is algebraically a group in which inversion is continuous. In what follows it is convenient to denote the pointwise sum of two functions g and h into a semigroup by g + h, i.e., if S is a semigroup and $g, h: Y \to S$, then $(g + h)(y) = g(y) + h(y)$ for all $y \in Y$.

(4.7) Proposition. Let X be a topological space, G be a topological group, $\mu: X^n \to X$ be a continuous function, $\nu: G^n \to G$ be a morphic n-mean on G and $f_0: X \to G$ be a continuous function. Then a continuous function $f: X^n \to G^n$ is such that the diagram



is analytic

iff $f = \Delta f_0\mu + g$ for some continuous function g from X^n into the kernel of ν, where $\Delta: G \to G^n$ is the diagonal map.

Proof. Note first that condition (i) in the definition of an n-mean implies that $\nu\Delta$ is the identity map on G. If $f = \Delta f_0\mu + g$ with $g: X^n \to \text{Ker } (\nu)$, then

$$\begin{aligned} \nu f &= \nu(\Delta f_0\mu + g) = \nu\Delta f_0\mu + \nu g \\ &= f_0\mu + 0 = f_0\mu \end{aligned}$$

since $\nu\Delta$ is the identity map on G.

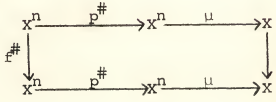
Conversely, if $\nu f = f_0\mu$ and if we define $g: X^n \to G^n$ by $g = f - \Delta f_0\mu$, then g is clearly continuous and the image of g is contained in the kernel of ν since

$$\nu g = \nu f - \nu\Delta f_0\mu = \nu f - f_0\mu = 0.$$

A modification of the following proposition generalizes a result of Kotz in [15]. It shows what happens to the homomorphisms of the n-mean structure when one "warps" the space. Note that there is no assumption that the mean is related to a semigroup structure on the space X.

(4.7) Proposition. If μ is an n-mean on a space X and $p: X \to X$ is continuous and onto and if we let E denote the set of continuous homomorphisms of the n-mean structure

on X into itself and E_p denote the set of those continuous functions $f: X \to X$ such that $\mu p^{\#} f^{\#} = f\mu p^{\#}$, i.e., such that the diagram



is analytic,

then $E_p = E \cap \{f: X \to X \mid pf = fp\}$ and the set of fixed points of p is invariant under each member of E_p.

Proof. If $f \in E_p$, then for each $x \in X$ we have that

$$\begin{aligned} pf(x) &= \mu(pf(x), \ldots, pf(x)) = \mu p^{\#} f^{\#}(x, \ldots, x) \\ &= f\mu p^{\#}(x, \ldots, x) = f\mu(p(x), \ldots, p(x)) \\ &= fp(x) \end{aligned}$$

so that $pf = fp$. Since p is onto, $p^{\#}$ is onto so if $x \in X^n$ we may choose $y \in X^n$ such that $p^{\#}(y) = x$ and hence

$$f\mu(x) = f\mu p^{\#}(y) = \mu p^{\#} f^{\#}(y) = \mu f^{\#} p^{\#}(y) = \mu f^{\#}(x),$$

i.e., $f \in E$.

If, on the other hand, $f \in E$ and $pf = fp$, then $p^{\#} f^{\#} = f^{\#} p^{\#}$ and we have that $\mu p^{\#} f^{\#} = \mu f^{\#} p^{\#} = f\mu p^{\#}$ and hence $f \in E_p$.

The last assertion of the theorem follows easily from the fact that each member of E_p commutes with p.

CHAPTER V

MEANS ON A CONTINUUM CONTAINING AN OPEN, DENSE HALF-LINE

In Aumann [2], Eckmann [6] and Eckmann, Ganea and Hilton [7], it is shown that there are wide classes of spaces which do not admit the structure of a mede. However, none of their methods apply, for example, to the space consisting of the curve $y = \sin \frac{1}{x}$, $0 < x \leq 1$, together with its limit continuum. It has been conjectured by A.D. Wallace that this space admits no mean but the question of whether it does remains open. In this chapter we present certain results concerning the admissibility of the structure of a mede on a continuum containing an open, dense half-line. The action of a semigroup on such a space has been studied by Day and Wallace [5] and, indeed, many of our techniques were suggested by their work.

We recall that if A and B are subsets of a topological groupoid then

$$AB = \{ab \mid a \in A \text{ and } b \in B\},$$

where ab denotes the value of the groupoid multiplication at (a,b), and we let

$$A^{(-1)}B = \{x \mid Ax \cap B \neq \emptyset\} \text{ and}$$
$$A^{[-1]}B = \{x \mid Ax \subset B\}$$

and, dually,

$$BA^{(-1)} = \{x \mid xA \cap B \neq \emptyset\} \text{ and}$$
$$BA^{[-1]} = \{x \mid xA \subset B\}.$$

It should be noted that if A is a singleton set, then $A^{(-1)}B = A^{[-1]}B$.

(5.1) Lemma. If A and B are subsets of a topological groupoid, then

(i) $A^{[-1]}B$ is open if A is compact and B is open, and dually.

(ii) $A^{(-1)}B$ is open if B is open, and dually.

(iii) $\{x \mid A \subset Ax\}$ is closed if A is compact, and dually.

Proof. (i) If $x \in A^{[-1]}B$, then $Ax \subset B$. But since A is compact, it follows from the continuity of the groupoid multiplication that there is an open set V about x such that $AV \subset B$. Hence $x \in V \subset A^{[-1]}B$ so that $A^{[-1]}B$ is open.

(ii) We have that $A^{(-1)}B = \cup\{a^{(-1)}B \mid a \in A\}$ $= \cup\{a^{[-1]}B \mid a \in A\}$ and each of $a^{[-1]}B$ is open by (i).

(iii) Since $\{x \mid A \subset Ax\} = \cap\{A^{(-1)}a \mid a \in A\}$

it suffices to show that $A^{(-1)}a$ is closed for each $a \in A$. If $x \notin A^{(-1)}a$, then $a \notin Ax$ and hence $Ax \subset X \setminus \{a\}$ which is open. Then, since A is compact, there is an open set V about x such that $AV \subset X \setminus \{a\}$ so that $x \in V \subset X \setminus A^{(-1)}a$. It follows that $A^{(-1)}a$ is closed.

Throughout this chapter we suppose that M is a continuum containing an open, dense half-line W whose complement C is non-degenerate and we let P be a property of topological spaces which is a continuous invariant on continua and which M does not possess.

(5.2) Lemma.

(i) C is a C-set (i.e., any continuum which intersects C but is not contained in C, must contain C.) and hence is connected.

(ii) Any subcontinuum of M which has property P and which intersects C must be contained in C.

Proof. (i) It is easy to see that C is a C-set and the fact that C is connected follows from lemma 1 of [22].

(ii) If, on the contrary, K is a subcontinuum of M having property P and intersecting both C and W, then by part (i) $C \subset K$ and hence, since $K \cap W \neq \square$, $K \overset{t}{=} M$. But since P is a continuous invariant on continua, this implies that M has property P contrary to our assumption.

The next theorem shows that the multiplication of an idempotent topological groupoid on M must "spread" near C.

(5.3) Theorem. If M is an idempotent topological groupoid, then W is a subgroupoid of M and either $C \subset Cx$ for all $x \in C$ or $C \subset xC$ for all $x \in C$, with $C \subset Cx$ for all $x \in C$ whenever either C or W is a right ideal of M and $C \subset xC$ for all $x \in C$ whenever either C or W is a left ideal of M. If, in addition, C has property P, then C is a subgroupoid of M, either C is a right ideal or W is a left ideal of M and, dually, either C is a left ideal or W is a right ideal of M.

Proof. The theorem is contained in the following series of assertions. The proofs of the primed assertions, which are dual to the unprimed ones, are omitted.

(i) W is a subgroupoid of M. We know that W^2 is arcwise connected since W is and that $W \subset W^2$ from the idempotency so that observing that W is an arc-component of M it follows that $W^2 = W$.

(ii) $M = W^{[-1]}W \cup W^{[-1]}C$. Since W is an arc-component of M and Wx is arcwise connected for each $x \in M$, we must have that, for each $x \in M$, either $Wx \subset W$ or $Wx \subset C$. This shows that $M \subset W^{[-1]}W \cup W^{[-1]}C$ and the reverse containment is trivial.

(ii)' $M = WW^{[-1]} \cup CW^{[-1]}$.

(iii) If $C \subset W^{[-1]}W$, then $C \subset Cx$ for all $x \in C$ and if $C \neq W^{[-1]}W$, then $C \subset xC$ for all $x \in M$. First, suppose that $C \subset W^{[-1]}W$ and let $x \in C$. Then Mx is a continuum intersecting C, since $x \in Mx$, but not contained in C, because $x \in C \subset W^{[-1]}W$ and hence $Wx \subset W \cap Mx$. It follows from lemma (5.2 (i)) that $C \subset Mx$ and since $Wx \subset W$, we must have that $C \subset Cx$.

Next, suppose that $C \neq W^{[-1]}W$ so that, in view of (i), we have that $C \cap W^{[-1]}C \neq \square$. If we choose $x_0 \in C \cap W^{[-1]}C$, then for $y \in W$, yM is a continuum which intersects C, because $yx_0 \in yC \cap Wx_0 \subset yM \cap C$, but which is not contained in C, since $y \in yM$, so that from lemma (5.2 (i)) we have that $C \subset yM$ and hence $C \subset yC$, since $yW \subset W$. We have shown that $W \subset \{x \mid C \subset xC\}$. But, by lemma (5.1 (iii)), $\{x \mid C \subset xC\}$ is closed from which it follows that $M = W^* \subset \{x \mid C \subset xC\}$.

(iii)' If $C \subset WW^{[-1]}$, then $C \subset xC$ for all $x \in C$ and if $C \neq WW^{[-1]}$, then $C \subset Cx$ for all $x \in M$.

(iv) $C \subset W^{[-1]}W$ iff W is a right ideal of M and $C \subset W^{[-1]}C$ iff C is a left ideal of M.

It is clear that $C \subset W^{[-1]}W$ iff $WC \subset W$, but since

$W^2 \subset W$, we have that $WC \subset W$ is equivalent to $WM \subset W$. Similarly, $C \subset W^{[-1]}C$ iff $WC \subset C$ which, in turn, is equivalent to $MC \subset C$, since C is closed and $W^* = M$.

(iv)' $C \subset WW^{[-1]}$ iff W is a left ideal of M and $C \subset CW^{[-1]}$ iff C is a right ideal of M.

(v) $C \subset Cx$ for all $x \in C$, whenever either C or W is a right ideal of M and $C \subset xC$ for all $x \in C$, whenever either C or W is a left ideal of M. Observing that $C \subset W^{[-1]}C$ implies that $C \neq W^{[-1]}W$ (and that $C \subset CW^{[-1]}$ implies that $C \neq WW^{[-1]}$) we see that (v) follows from combining (iii), (iii)', (iv) and (iv)'.

(vi) If C has property P, then C is a subgroupoid of M. If $x \in C$, then $x \in xC \cap C$ and C has property P so, by lemma (5.2 (ii)), $xC \subset C$, i.e., $C^2 \subset C$.

(vii) If C has property P, then either $C \subset W^{[-1]}W$ or $C \subset W^{[-1]}C$. If $C \neq W^{[-1]}W$ with, say, $x_0 \in C \setminus W^{[-1]}W$, then in view of (i) we have that $Wx_0 \subset C$ so that, for each $y \in W$, $yx_0 \in yC \cap C$ and hence from the fact that yC has property P and from (4.2 (ii)) it follows that $yC \subset C$. This shows that $WC \subset C$ and hence $C \subset W^{[-1]}C$.

(vii)' If C has property P, then either $C \subset WW^{[-1]}$ or $C \subset CW^{[-1]}$.

An immediate consequence of the preceding theorem is the following

(5.4) Corollary. If M is a mede, then $C \subset Cx$ for all $x \in C$, W is a submede of M and if, in addition, C has property P, then $C = Cx$ for all $x \in C$ and either C or W is an ideal of M.

In the case of an idempotent semigroup, theorem (5.3) reduces to the following corollary. A topological semilattice is an associative mede.

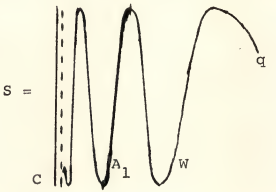
(5.5) Corollary. If M is an idempotent semigroup, then C is a subsemigroup of M such that either $xy = x$ for all $x, y \in C$ or $xy = y$ for all $x, y \in C$. Hence, M admits no topological semilattice.

Proof. If M is an idempotent semigroup, then, from theorem (5.3), we have that either $C \subset Cx$ for all $x \in C$ or $C \subset xC$ for all $x \in C$ and hence, since each element of C is idempotent, C consists of either all right units or all left units of C. For, supposing $C \subset Cx$ for all $x \in C$, if $x, y \in C$, then $x \in C \subset Cy$ and hence $x = zy$ for some $z \in C$ from which it follows that $xy = (zy)y = zy^2 = zy = x$. The dual assertion follows similarly.

If M were a topological semilattice, then from the above, each element of C would be a unit for C, which

would imply that C is degenerate contrary to our assumption.

Let S be the subcontinuum of the plane consisting of the union of the sets $C = \{(r,s) \mid r = 0 \text{ and } -1 \leq s \leq 1\}$ and $W = \{(r,s) \mid s = \sin \frac{1}{r} \text{ and } 0 < r \leq 1\}$ and, for each positive integer K, let A_k be the subcontinuum of S given by $A_k = \{(r,s) \in S \mid \frac{2}{(4k+3)\pi} \leq r \leq \frac{2}{(4k+1)\pi}\}$.



It will be convenient to denote the subcontinuum of S irreducible between A and B by [A,B], where A and B are subcontinua of S and to let $\langle A,B] = [A,B] \setminus A$, etc.

The remainder of this chapter is devoted to showing that any autodistributive (and hence any medial) mede on the sinusoidal continuum described above must have the limit continuum as an ideal. An example of a continuum admitting no autodistributive mede is also given. We first prove a lemma.

(5.6) Lemma. If A is a submede of an autodistributive mede M, then Ax is a submede for each $x \in M$.

Proof. We have only to observe that if A is a submede of M, then the autodistributivity of M implies that

$$(Ax)(Ax) = AA \cdot x = Ax$$

for all $x \in M$.

A nodal set in a continuum is a proper, non-void subcontinuum having a one point boundary.

(5.7) Proposition. If S is an autodistributive mede then C is an ideal of S.

Proof. In view of corollary (5.4) (with M = S and P = local connectivity) we have only to show that W is not an ideal of S. To this end we suppose, on the contrary, that W is an ideal of S and note that, for each $y \in W$, $Sy = [r_y, t_y]$ for some r_y, $t_y \in W$ and $Cy = [e_y, f_y]$ for some e_y, $f_y \in W$, and we set $N_0 = [C, r_q]$ where q is the endpoint of S.

We now prove several assertions whose object is to show that the mean is internal on C, in the sense that if $x, y \in C$, then $xy \in [x,y]$.

(i) $N_0 \cap W \subset \{r_y \mid y \in W\} \cap \{f_y \mid y \in W\}$. If we define $g: W \to W$ by $g(y) = r_y$, then g is continuous, for if we let $\langle u,v \rangle$ be an open interval contained in W, let $U_1 = [C,V\rangle$, and let $U_2 = \langle u,q]$, then U_1 and U_2 are both

open and, since S is compact, so are $S^{(-1)}U_1$ and $S^{[-1]}U_2$. But W is open and

$$g^{-1}(\langle u,v\rangle) = S^{(-1)}U_1 \cap S^{[-1]}U_2 \cap W$$

from which follows the continuity of g. If one defines $h: W \to W$ by $h(y) = f_y$ then one shows h continuous in a similar manner. It follows that each of g(W) and h(W) is connected.

Now let $y \in N_0 \cap W$ and observe that $S^{(-1)}[C,y\rangle$ and $C^{[-1]}[C,y\rangle$ are open, the latter because C is compact, and that $C \subset C^{[-1]}[C,y\rangle \cap S^{(-1)}[C,y\rangle$ since, for $x \in C$, we have that $Cx = C \subset [C,y\rangle$ and $x \in Sx \cap C \subset Sx \cap [C,y\rangle$. We may then choose an element $x \in C^{[-1]}[C,y\rangle \cap S^{(-1)}[C,y\rangle \cap W$ so that, on the one hand, $Cx \subset [C,y\rangle$ and hence $y \in [f_x,r_q] \subset [f_x,f_q] \subset h(W) = \{f_y \mid y \in W\}$ (h(W) is connected!) and, on the other hand, $Sx \cap [C,y\rangle \neq \square$ so that $y \in [r_x,r_q] \subset g(W) = \{r_y \mid y \in W\}$.

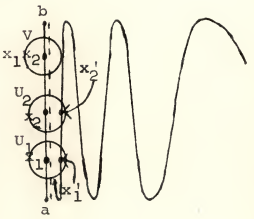
(ii) *If* $h: W \to W$ *is defined by* $h(y) = f_y$ *and if* N *is a nodal set containing* C, *then there is a nodal set* N' *about* C *contained in* N_0 *and disjoint from* $h([q_0,q])$, *where* $q_0 = F(N)$, *which must then satisfy* $N' \cap W \subset h(N \cap W)$. The function h is continuous, as was shown in the proof of (i), so that $h([q_0,q])$ is compact and since it is contained in W it is clear that we may choose a nodal set N' about C disjoint from $h([q_0,q])$ and contained in N_0. From (i)

we have that $N_0 \cap W \subset h(W)$, and hence $N' \cap W \subset N_0 \cap W \subset h(W)$ $= h(N \cap W) \cup h([q_0,q])$ from which it follows that $N' \cap W \subset h(N \cap W)$ since $(N' \cap W) \cap h([q_0,q]) \subset N' \cap h([q_0,q])$ $= \square$.

(iii) *If x_1, $x_2 \in C$ and U_1 and U_2 are open sets about x_1 and x_2, respectively, then there exists a nodal set N of S such that $C \subset N \subset C^{(-1)}U_1 \cap C^{(-1)}U_2$.* Each of $C^{(-1)}U_1$ and $C^{(-1)}U_2$ are open and $C \subset C^{(-1)}U_1 \cap C^{(-1)}U_2$ because, in view of corollary (5.4), $\{x_1, x_2\} \subset C = Cx$ for all $x \in C$. But an open set about C must contain a nodal set about C.

(iv) *The mean is internal on C.* Suppose that $x_1x_2 \notin [x_1,x_2]$ for some x_1, $x_2 \in C$ and assume that $\pi_2x_1 \leq \pi_2x_2$, where π_2 is the second projection in the plane. Let V be a spherical (using the usual metric of the plane) neighborhood of x_1x_2 such that $V^* \cap \{x_1, x_2\} = \square$ and, using the continuity of the mean, let U_1 and U_2 be spherical neighborhoods of x_1 and x_2 respectively, such that $U_1U_2 \subset V$ and $(U_1 \cup U_2) \cap V = \square$, the latter being possible because of our choice of V. Using (iii), we let N be a nodal set about C such that $N \subset N_0$ and $N \subset C^{(-1)}U_2 \cap C^{(-1)}V$ and then, using (ii), we let N' be a nodal set containing C such that $N' \cap W \subset h(N \cap W)$. Let k be so large that $A_k \subset N'$ and $A_k \cap U_1 \neq \square \neq A_k \cap U_2$ and choose $x_1' \in A_k \cap U_1$

and $x_2' \in A_k \cap U_2$.

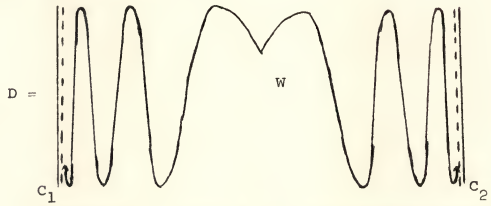


From (i) we have that $x_1' = r_{y_1}$ for some $y_1 \in W$, $x_2' = f_{y_2}$ for some $y_2 \in W$ and because $x_2' \in A_k \subset N' \cap W \subset h(N \cap W)$ we may suppose that $y_2 \in N$ so that $Cy_2 \cap U_2 \neq \square \neq Cy_2 \cap V$. It follows that $r_{y_1} = x_1' \in Cy_2 = [e_{y_2}, f_{y_2}]$ and hence, recalling that $f_{y_2} = x_2' \in A_k \subset N_0$ we have that $Cy_2 \cap Sy_1 = [r_{y_1}, f_{y_2}] = [x_1', x_2']$ so that since each of Cy_2 and Sy_1 are submedes of M, so is $[x_1', x_2']$, from which we have that $x_1'x_2' \in [x_1', x_2']$. But $x_1'x_2' \subset U_1U_2 \subset V$ while $V \cap [x_1', x_2'] = \square$, a contradiction.

Returning to the proof of the proposition and letting a and b denote the endpoints of C, we note that by corollary (5.4), $a \in C = Cb$ so that $a = xb$ for some $x \in C$. But, by (iv), the mean is internal on C and hence $x = a$, i.e., $ab = a$. In a similar fashion we obtain $ab = b$ and hence $a = b$, a contradiction. Therefore

our assumption that W is an ideal of S is false.

We next let $D = S \cup \{(r,s) \mid (2 - r,s) \in S\}$ and again employ the notation [A,B] for the continuum irreducible between A and B, where A and B are subcontinua of D. Observe that $C_1 = \{(r,s) \in D \mid r = 0\}$, $C_2 = \{(r,s) \in D \mid r = 2\}$ and $W = \{(r,s) \in D \mid 0 < r < 2\}$ are the arc-components of D.



(5.8) Proposition. D admits no autodistributive mean.

Proof. Suppose, on the contrary that D is an autodistributive mede. From the fact that each of C_1, C_2 and W are the arc components of D we have that they are each submedes of D.

We establish a contradiction by the following series of assertions.

(i) Either $WC_1 \subset C_1$ or $WC_1 \subset W$ and, dually, either $WC_2 \subset C_2$ or $WC_2 \subset W$. Since WC_1 is arcwise connected and the arc-components of D are C_1, C_2 and W, in order to

show that $WC_1 \subset C_1$ or $WC_1 \subset W$, it suffices to show that $WC_1 \neq C_2$. We know that $C_1^{[-1]}(M \setminus C_2)$ is open and contains C_1 because C_1 is compact and $C_1^2 \subset C_1$ and hence $W \cap C_1^{[-1]}(M \setminus C_2) \neq \square$ from which we obtain that $C_1W \cap (M \setminus C_2) \supset C_1[W \cap C_1^{[-1]}(M \setminus C_2)] \neq \square$, i.e., $WC_1 = C_1W \neq C_2$. The dual assertion follows similarly.

(ii) Either $WC_1 \subset W$ or $WC_2 \subset W$. If this were not true, then, in view of (i), we must have $WC_1 \subset C_1$ and $WC_2 \subset C_2$ and hence $DC_1 = W^*C_1 \subset C_1$ and $DC_2 = W^*C_2 \subset C_2$ from which follows the impossible conclusion that both $C_1C_2 \subset DC_2 \subset C_2$ and $C_1C_2 = C_2C_1 \subset DC_1 \subset C_1$.

In view of the symmetry, we suppose for the remainder of the proof that $WC_2 \subset W$ and consider the two possibilities $WC_1 \subset C_1$ (and hence $DC_1 \subset C_1$) and $WC_1 \subset W$.

(iii) If $DC_1 \subset C_1$, then $C_2 = C_2x$ for all $x \in C_2$. If $x \in C_2$, then Dx is a subcontinuum of D intersecting C_2, since $C_2x \subset C_2$, but not contained in C_2, since $Wx \subset WC_2 \subset W$ and hence, since C_2 is a C-set, we have that $C_2 \subset Dx$. It follows that $C_2 \subset C_2x$ (and hence $C_2 = C_2x$) because, in addition to $Wx \subset W$, we have that $C_1x \subset C_1D = DC_1 \subset C$.

(iv) If $DC_1 \subset C_1$, then, for each $y \in W$, $Dy = [C_1, s_y]$ for some $s_y \in W$ and $W = \{s_y \mid y \in W\}$.

It is clear that if $y \in W$, then $Dy = [C_1, s_y]$ for some $s_y \in W$ and if we define $f: W \to W$ by $f(y) = s_y$, one shows that f is continuous in a manner similar to that in the proof of assertion (i) of proposition (5.7) and hence it follows that $f(W)$ is connected.

If $y \in W$, then $D^{(-1)}\langle y, C_2]$ is open and contains C_2 ($x \in C_2 \Rightarrow x \in Dx \cap C_2 \subset Dx \cap \langle y, C_2]$) and, because D is compact, $D^{[-1]}[C_1, y\rangle$ is open and contains C_1 ($DC_1 \subset C_1 \subset [C_1, y\rangle$), so if we let $y_1 \in D^{[-1]}[C_1, y\rangle \cap W$ and $y_2 \in D^{(-1)}\langle y, C_2] \cap W$, then $y \in [s_{y_1}, s_{y_2}] \subset f(W)$ since $f(W)$ is connected. We have therefore shown that $W \subset f(W) = \{s_y \mid y \in W\}$.

(v) If $WC_1 \subset W$ and if $x_1 \in C_1$, $x_2 \in C_2$, then either $Dx_1 \cap C_2 = \square$ or $Dx_2 \cap C_1 = \square$. Since each of C_1 and C_2 are C-sets and since $Dx_1 \cap C_1 \neq \square \neq Dx_2 \cap C_2$ ($x_1 \in Dx_1$, $x_2 \in Dx_2$), if we suppose that the assertion is not true we must have that $Dx_1 = D$ and $Dx_2 = D$. Each of C_1x_1, Wx_1 and C_2x_1 are arcwise connected with $C_1x_1 \subset C_1$ and $Wx_1 \subset WC_1 \subset W$ so that since $D = C_1x_1 \cup Wx_1 \cup C_2x_1$ and C_1, C_2 and W are the arc-components of D, it follows that $C_2 = C_2x_1$. We then have $x_1x_2 \in C_2$ and a dual argument shows that $x_1x_2 \in C_1$ which is impossible.

If $DC_1 \subset C_1$, by utilizing (iv) and our assumption that $WC_2 \subset W$ one can show as in the proof of proposition

(5.7) that the mean is internal on C_2 which, again as in the proof of proposition (5.7), will then contradict (iii).

If $WC_1 \subset W$ (and $WC_2 \subset W$), then, in view of (v) and the symmetry we may suppose that $Dx_1 \cap C_2 = \square$ with $x_1 \in C_1$. Then Dx_1 is a continuum contained in $C_1 \cup W$ which intersects both C_1 and W so that since C_1 is a C-set, it contains C_1 and hence is homeomorphic to the sinusoidal continuum S defined earlier. Furthermore, Dx_1 is a submede of D such that $(Dx_1 \cap W)C_1 \subset Dx_1 \cap W$ so that if we apply proposition (5.7) we are led to a contradiction.

Having shown that all possibilities lead to a contradiction, the proposition follows.

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BIOGRAPHICAL SKETCH

Kermit Neal Sigmon was born April 18, 1936, in Lincoln County, North Carolina. In May, 1954, he was graduated from Rock Springs High School in Denver, North Carolina. In May, 1958, he received the degree of Bachelor of Arts from Appalachian State Teachers College and in August, 1959, he received the degree of Master of Education from the University of North Carolina at Chapel Hill. From 1959 until 1963 he taught mathematics in the high schools of Charlotte, North Carolina. In 1963 he enrolled in the Graduate School of the University of Florida where until the present time he has served as a part time instructor in the Department of Mathematics while pursuing work toward the degree of Doctor of Philosophy.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 1966

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